RMSC 4003
Statistical Modeling in Financial Markets
Tutorial 5 Solution

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October 9, 2014

1 R and Excel Demonstration

1.1 Estimation of covariance matrix

Recall that to estimate the variance of a random variable X, we can use the sample variance (STAT2006) (which is an unbiased estimator of the variance):

$$S^2 := \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2,$$

where $\overline{X} := \sum_{i=1}^{n} X_i/n$. Similarly, to estimate the covariance matrix of a random vector, we can use the sample covariance matrix (which is an unbiased estimator of the covariance matrix):

$$\widehat{\Sigma} := \frac{1}{n-1}\sum_{i=1}^{n}(\boldsymbol{X_i} - \overline{\boldsymbol{X}}))(\boldsymbol{X_i} - \overline{\boldsymbol{X}}))^T,$$

where $\overline{\boldsymbol{X}} = (\overline{X}_1, \dots, \overline{X}_n)^T$ and $\boldsymbol{X}_i = (X_{1i}, \dots, X_{ni})^T$. The above form is equivalent to

$$\widehat{\Sigma} := \frac{1}{n-1}\begin{pmatrix} \sum_{i=1}^{n}(X_{1i} - \overline{X}_1)^2 & \sum_{i=1}^{n}(X_{1i} - \overline{X}_1)(X_{2i} - \overline{X}_2) & \dots & \sum_{i=1}^{n}(X_{1i} - \overline{X}_1)(X_{ni} - \overline{X}_n) \\ \vdots & \ddots & \ddots & \vdots \\ \sum_{i=1}^{n}(X_{ni} - \overline{X}_n)(X_{1i} - \overline{X}_1) & \sum_{i=1}^{n}(X_{ni} - \overline{X}_n)(X_{2i} - \overline{X}_2) & \dots & \sum_{i=1}^{n}(X_{ni} - \overline{X}_n)^2 \end{pmatrix}.$$

1.2 Efficient Frontier without risk-free asset

To plot the efficient frontier, you should only plot the points above the global minimum variance point. For 2 assets, you can find out the weight that minimizes the variance of the portfolio easily. Indeed, let $r_P = \alpha r_H + (1-\alpha) r_S$. Then

$$\sigma_P^2 = \alpha^2\sigma_H^2 + (1-\alpha)^2\sigma_S^2 + 2\alpha(1-\alpha)\sigma_{HS}$$

and

$$\frac{d\sigma_P^2}{d\alpha} = 2\alpha\sigma_H^2 - 2(1-\alpha)\sigma_S^2 + (2-4\alpha)\sigma_{HS}.$$

Setting the derivative to 0, we have

$$\alpha_{\min} = \frac{\sigma_S^2 - \sigma_{HS}}{\sigma_H^2 + \sigma_S^2 - 2\sigma_{HS}}.$$

In general (for n assets), you can find out the global minimum variance portfolio by the method in Section 2 of Tutorial 3. For actual implementation, refer to the R code and the Excel file (for homework II).

1.3 Efficient Frontier with risk-free asset

In Homework II Question 4, the risk-free rate is higher than the mean of the global minimum variance portfolio. This makes the tangency portfolio appears on the lower part of the minimum set (which is a hyperbola in general). If you look at the graph on P.25 of the lecture notes, you should notice that there are actually two lines with intercept $(0, r_f)$. Suppose we have a portfolio with $r_P = \alpha r_T + (1-\alpha)r_f$. Then

$$\begin{aligned} \mu_P &= \alpha\mu_T + (1-\alpha)r_f \\ \sigma_P^2 &= \alpha^2\sigma_T^2. \end{aligned}$$

If $\alpha \geq 0$, $\sigma = \alpha\sigma_T$; if $\alpha < 0$, $\sigma_P = -\alpha\sigma_T$ (be careful when you take the square root). Therefore, if $\alpha \geq 0$,

$$\begin{aligned} \mu_P &= \frac{\sigma_P}{\sigma_T}\mu_T + \frac{\sigma_T - \sigma_P}{\sigma_T} r_f \\ &= \frac{\mu_T - r_f}{\sigma_T}\sigma_P + r_f. \end{aligned}$$

If $\alpha < 0$,

$$\begin{aligned} \mu_P &= -\frac{\sigma_P}{\sigma_T}\mu_T + \frac{\sigma_T + \sigma_P}{\sigma_T} r_f \\ &= -\frac{\mu_T - r_f}{\sigma_T}\sigma_P + r_f. \end{aligned}$$

These are the equations of the two lines. As the two lines have slopes with opposite value and with the same absolute value, they are symmetric about the line $\mu_P = r_f$.

Remark 1.1. If $r_P = \sum_{i=1}^n w_i r_i$, then $\sigma_P^2 = \sum_{i=1}^n \sum_{j=1}^n w_i w_j \sigma_{ij} = \boldsymbol{w}^T \Sigma \boldsymbol{w}$ where $\boldsymbol{w} = (w_1, \ldots, w_n)^T$ and Σ is the covariance matrix.

2 Market Model

Denote r_i to be the return of asset i and r_M to be the market return. Market model:

$$r_i = \alpha_i + \beta_{iM} r_M + \varepsilon_i, \tag{1}$$

where $\operatorname{Cov}(\varepsilon_i, r_M) = 0$. Note that $\beta_{iM} = \dfrac{\operatorname{Cov}(r_i, r_M)}{\operatorname{Var}(r_M)}$ since

$$\operatorname{Cov}(r_i, r_M) = \operatorname{Cov}(\alpha_i + \beta_{iM} r_M + \varepsilon_i, r_M) = \beta_{iM}\operatorname{Cov}(r_M, r_M) + \operatorname{Cov}(\varepsilon_i, r_M) = \beta_{iM}\operatorname{Var}(r_M).$$

Taking expectation and variance in (1), we have

$$\mu_i = \alpha_i + \beta_{iM}\mu_M \text{ and } \sigma_i^2 = \beta_{iM}^2\sigma_M^2 + \sigma_{\varepsilon i}^2,$$

where σ_M^2 denotes the market (systemic) risk, $\beta_{iM}^2\sigma_M^2$ represents the market risk of security i and $\sigma_{\varepsilon i}^2$ denotes the unique (unsystematic or idiosyncratic) risk.

2.1 Estimation of Beta

Consider the market model of asset i over time:

$$r_{it} = \alpha_{it} + \beta_{iM} r_{Mt} + \varepsilon_{it} \quad t = 1, \dots, T.$$

We can estimate β_{iM} by using linear regression (STAT3008). Ordinary least square (OLS) of r_{it} against r_{Mt} gives

$$\widehat{\beta}_{iM} = \frac{\sum_{t=1}^{T}(r_{it} - \overline{r}_i)(r_{Mt} - \overline{r}_M)}{\sum_{t=1}^{T}(r_{Mt} - \overline{r}_M)^2},$$

where $\overline{r}_i := \frac{1}{T}\sum_{t=1}^{T} r_{it}$ and $\overline{r}_M := \frac{1}{T}\sum_{t=1}^{T} r_{Mt}$. R code for linear regression:

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reg = lm(stock_return ~ market_return)
summary(reg)
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3 Capital Asset Pricing Model

Definition 3.1 (Market Portfolio). *The market portfolio is a portfolio consisting of all securities, where the proportion invested in each security corresponds to its relative market value, which is defined as*

$$\textit{Relative market value of a security} = \frac{\textit{Aggregate Market Value of the Security}}{\textit{Aggregate Market Value of All Securities}}.$$

Under the perfect market conditions, every investor will purchase the same single fund of risky assets according to one-fund theorem. Hence, the one fund must be the market portfolio.

- If both risk-free lending and borrowing are allowed, the efficient frontier is a straight line, called the Capital Market Line (CML):

$$\mu_P = r_f + \frac{\mu_M - r_f}{\sigma_M}\sigma_P.$$

- Price of risk = slope of CML = K = $\frac{\mu_M - r_f}{\sigma_M}$
- For ANY portfolio, Sharpe ratio (S) := $\frac{\mu - r_f}{\sigma}$

Theorem 3.1 (CAPM). *If the market portfolio is efficient, the expected return of any asset r_i must satisfy*

$$\mu_i - r_f = \beta_i(\mu_M - r_f),$$

where $\beta_i = \frac{\sigma_{iM}}{\sigma_M^2}$ and $\sigma_{iM} = \mathrm{Cov}(r_i, r_M)$. The above equation is known as the Security Market Line (SML).

- Given μ_M, σ_M, r_f, we know that $\mu_i - r_f \propto \sigma_{iM}$ from SML;
- CML: plotting μ against σ;
- SML: plotting μ against β.

Example 3.1. If $r_p = \sum_{i=1}^{N} w_i r_i$, show that $\beta_p = \sum_{i=1}^{N} w_i \beta_i$.

Proof.

$$\beta_P = \frac{\mathrm{Cov}(r_P, r_M)}{\mathrm{Var}(r_M)} = \frac{\sum_{i=1}^{n} w_i \,\mathrm{Cov}(r_i, r_M)}{\mathrm{Var}(r_M)} = \sum_{i=1}^{n} w_i \frac{\mathrm{Cov}(r_i, r_M)}{\mathrm{Var}(r_M)} = \sum_{i=1}^{n} w_i \beta_i.$$

□

Example 3.2. Suppose that we write the market model as

$$r_i - r_f = \beta_i (r_M - r_f) + \varepsilon_i. \tag{2}$$

Assume CAPM holds, show that

$$\sigma_i^2 = \beta_i^2 \sigma_M^2 + \sigma_{\varepsilon i}^2 = \text{Systematic risk} + \text{Idiosyncratic risk}.$$

Show also that asset on the CML contains systematic risk but not idiosyncratic risk.

Solution. Taking expectation on both sides of (2) and assume CAPM holds, $\mathrm{E}(\varepsilon_i) = 0$. Taking covariance with r_M on both sides yield

$$\sigma_{iM} = \beta_i \sigma_M^2 + \mathrm{Cov}(\varepsilon_i, r_M),$$

implying $\mathrm{Cov}(\varepsilon_i, r_M) = 0$ since $\beta_i = \sigma_{iM}/\sigma_M^2$. Thus,

$$\sigma_i^2 = \beta_i^2 \sigma_M^2 + \sigma_{\varepsilon i}^2 = \text{Systematic risk} + \text{Idiosyncratic risk}.$$

Suppose that there is an asset lying on the CML. The asset is efficient and can be expressed by a linear combination of r_M and r_f:

$$\begin{aligned} r_i &= w r_M + (1-w) r_f \\ r_i - r_f &= w(r_M - r_f). \end{aligned}$$

Taking expectation on both sides, we have $\mu_i - r_f = w(\mu_M - r_f)$. Thus, $w = \beta_i$. Hence,

$$\sigma_i^2 = \beta_i^2 \sigma_M^2 = \text{Systematic risk}.$$

Remark 3.1. Recall that $\beta_i = \dfrac{\sigma_{iM}}{\sigma_M^2}$. So β_i does not necessarily increase with volatility σ_i, although empirical results show that the two are highly correlated with each other.

Example 3.3. Consider the summary statistics of the annualized returns of two stocks following the CAPM:

Securities	Expected Return (%)	Standard deviation (%)	Correlation with r_M
Stock A	18	30	0.4
Stock B	30	50	0.88

Suppose the market portfolio has variance of 16%.

(a) What is the beta of each security?

(b) What is the expected return of the market portfolio?

(c) What are the equations for the Capital Market Line and the Security Market Line?

Solution. (a) Using the fact that $\beta_i = \frac{\sigma_{iM}}{\sigma_M^2} = \frac{\rho_{iM}\sigma_i\sigma_M}{\sigma^M} = \frac{\rho_{iM}\sigma_i}{\sigma_M}$,

$$\begin{aligned} \beta_A &= \rho_{AM}\sigma_A/\sigma_M = 0.4(0.3)/0.4 = 0.3 \\ \beta_B &= \rho_{BM}\sigma_B/\sigma_M = 0.88(0.5)/0.4 = 1.1. \end{aligned}$$

(b) Based on the expected returns and betas of any two stocks, we are able to back out the expected portfolio return and the risk-free rate as follows:

$$\begin{aligned} 0.18 - r_f &= 0.3(\mu_M - r_f) \\ 0.3 - r_f &= 1.1(\mu_M - r_f) \end{aligned}$$

So, $\mu_M = 0.285$ and $r_f = 0.135$.

(c) Price of risk $K = (\mu_M - r_f)/\sigma_M = 0.15/0.4 = 0.375$.

$$\begin{aligned} \text{CML} : \mu_P &= 0.135 + 0.375\sigma_P \\ \text{SML} : \mu_i &= 0.135 + 0.15\beta_i. \end{aligned}$$